Reprinted from
AUTOMATIC CONTROL®
Data Systems Engineering
trademark
Vol. 17 No. 4 November 1962

Considerations in the Application of Flexural Pivots

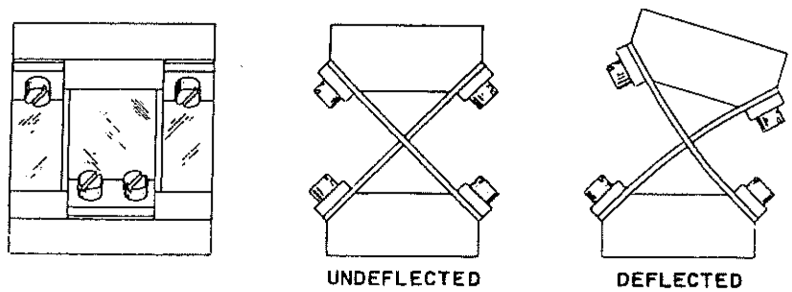

Fig. 1. Bisymmetrical flexural pivot.

Sensitivity, accuracy and repeatability are limited primarily by pivot friction and backlash where pivoted members are used to transfer or algebraically operate upon forces or displacements. Since many instruments, controls and measuring devices employ pivoted members the reduction of pivot friction and backlash is consequential to the reduction of their contributed error.

by **Henry Troeger,**
Chief Engineer, Advanced Design
The Bendix Corp., Utica Division
Utica, New York

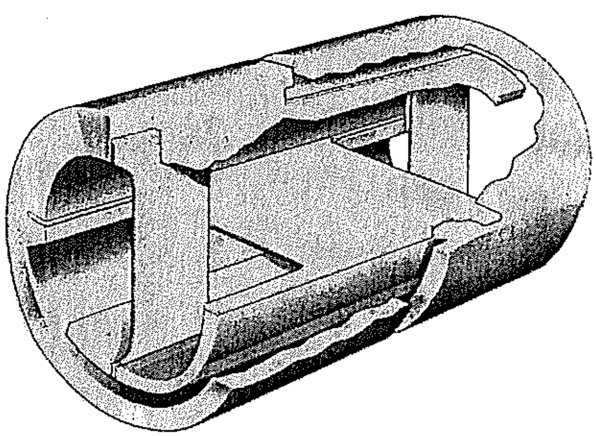
Fig. 2. New integrated, standardized cylindrical flexural pivot.

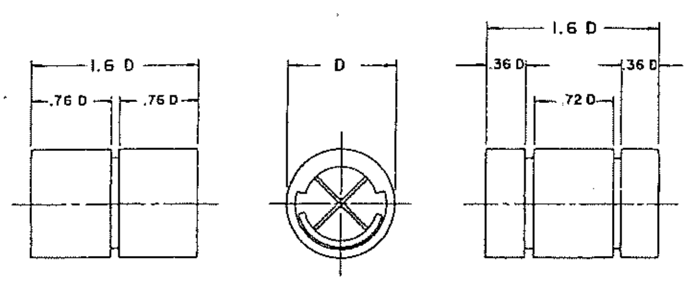

Fig. 3. Standard cylindrical flexural pivot proportions.

■ The bisymmetrical flexural pivot, Figure 1, is made by joining two members between which rotation is desired with relatively thin flat flexures positioned so their planes are normal to each other and the intersection of their planes is on the desired axis of rotation. Each flexure constrains the bending mode of the other thus being one major factor determining the centrode. Other major factors are radial load and radial spring rate.

The advantages of the flexural pivot are: the absence of rolling or coulomb friction and backlash, no requirement for lubrication, insensitivity to contamination, the ability to operate over a wide range of environmental conditions including vacuum, pressure, low and high temperature and radiation. Its limitations are: limited angular travel, torsional spring rate (often an advantage), the centrode and center shift, and, depending upon angular travel requirements and available space, load capacity.

Flexural pivots have had restricted use because, until recently, they have been tailored to each application and the number of pieces required made them relatively expensive to manufacture and assemble and difficult to service. Applications were curtailed by the complexity of mathematical determination of characteristics for particular operating conditions. Requirements for experimental verification of computable characteristics and for the determination of those characteristics best established by test, such as fatigue life, hysteresis and linearity, also discouraged use.

These objections are eliminated by the new, integrated, standardized and mass produced cylindrical flexural pivot shown in Figure 2*. The outer sleeves are shown partially broken away in order to show the internal configuration. Each of the two adjacent sleeves has an annular sector which projects into the other sleeve but does not touch it. The two sectors are joined by a set of flat crossed flexures. The sleeves, sectors and flexures are secured together by high temperature alloy brazing and the assembly is heat treated to spring temper the flexures. It is installed in round holes in the support and pivoted member. Two types are available: cantilever and double-end. The cantilever type has two adjacent concentric sleeves of the same diameter. Either sleeve may be mounted in a support and the adjacent sleeve mounted in the member to be pivotally supported. The double-end type is similar in internal construction but has three adjacent sleeves the outer two of which are internally secured together and are rotatable with respect to the center sleeve. This permits a bridge type support which allows higher radial loads and provides more rigid radial constraint than a cantilever type to which an offset load is applied.

These cylindrical flexural pivots are being manufactured in diameters of 0.125, 0.156, 0.1875, 0.250, 0.3125, 0.375, 0.500, 0.625, 0.750, 0.875, and 1.000 inches. Angular travel stops are at plus and minus 7.5, 15, and 30 degrees. Larger angular travel pivots have thinner springs and low torsional spring rates. Smaller angular travel pivots have thicker springs and high load carrying capacity. Proportions are shown in Figure 3. The flexures are AISI 420 corrosion resistant steel. Cylindrical flexural pivots made of constant modulus and other materials are being developed.

*FREE-FLEX® flexural pivot (patent applied for) manufactured by the Bendix Corporation, Utica Division, Utica, New York

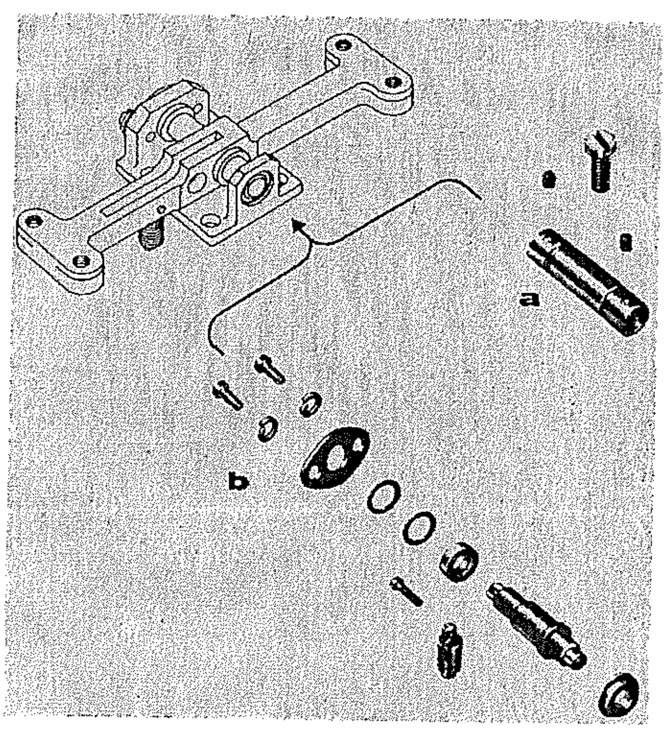

Fig. 4. Cylindrical flexural pivot (a) replaces previously used ball bearing pivot (b) eliminating eight parts in this air data sensor transducer.

These cylindrical flexural pivots have certain improved characteristics over conventional constructions utilizing flexures clamped by screws or rivets. The flexures are filleted at their joints thus providing stress relief and distribution of load over a large braze area. The high strength braze eliminates slippage between the mating joint surfaces thus precluding hysteresis due to joint slippage. The small radial dimension required for a brazed joint compared to a clamped one results in a compact unit. And, of course, quantity production permits an otherwise prohibitive investment in tooling and test equipment necessary for economical manufacture and determination of characteristics.

Illustrative of a cylindrical flexural pivot application is the air data sensor transducer* shown in Figure 4. This sensor has a sensitivity of three feet at 10,000 feet altitude. Previously used ball bearings required extreme cleanliness and care in assembly and adjustment. Substitution of a cylindrical flexural pivot eliminated eight parts and test rejections due to dirt or improper bearing adjustment, saved assembly time, permitted larger and fewer tolerances and resulted in improved peformance.

The following flexural pivot characteristics should be considered by the engineer and designer:

1. Adaptability
2. Center shift and centrode
3. Torsional spring rate
4. Angular travel
5. Linearity
6. Hysteresis
7. Radial and axial load capacity
8. Radial spring rate
9. Fatigue life
10. Environmental adaptability

Adaptability: Flexural pivots may be used where the combination of angular travel, load, spring rate and center shift requirements are acceptable. These characteristics, later discussed in detail, are interrelated to dimensions and material. Several methods of mounting the cylindrical flexural pivot are:

Split hole with clamping screw,
Press fit in hole split to limit pressing force,
Free fit in hoel with set screw,
Press fit in hole with tools which avoid stressing flexures.
Cement or solder in place

When two cantilever pivots are used to support an intermediate member, consideration must be given alignment to avoid introducing spurious stresses. Flexible supports that will permit self alignment of the pivot axes may be desireable.

Center shift and centrode: When a flexural pivot is angularly deflected the resulting geometry of the flexures determines the transposition of all points of the moving member. Of particular interest are the paths of two points: the geometric center and the instantaneous center.

*Manufactured by Eclipse-Pioneer Division, Bendix Corporation.

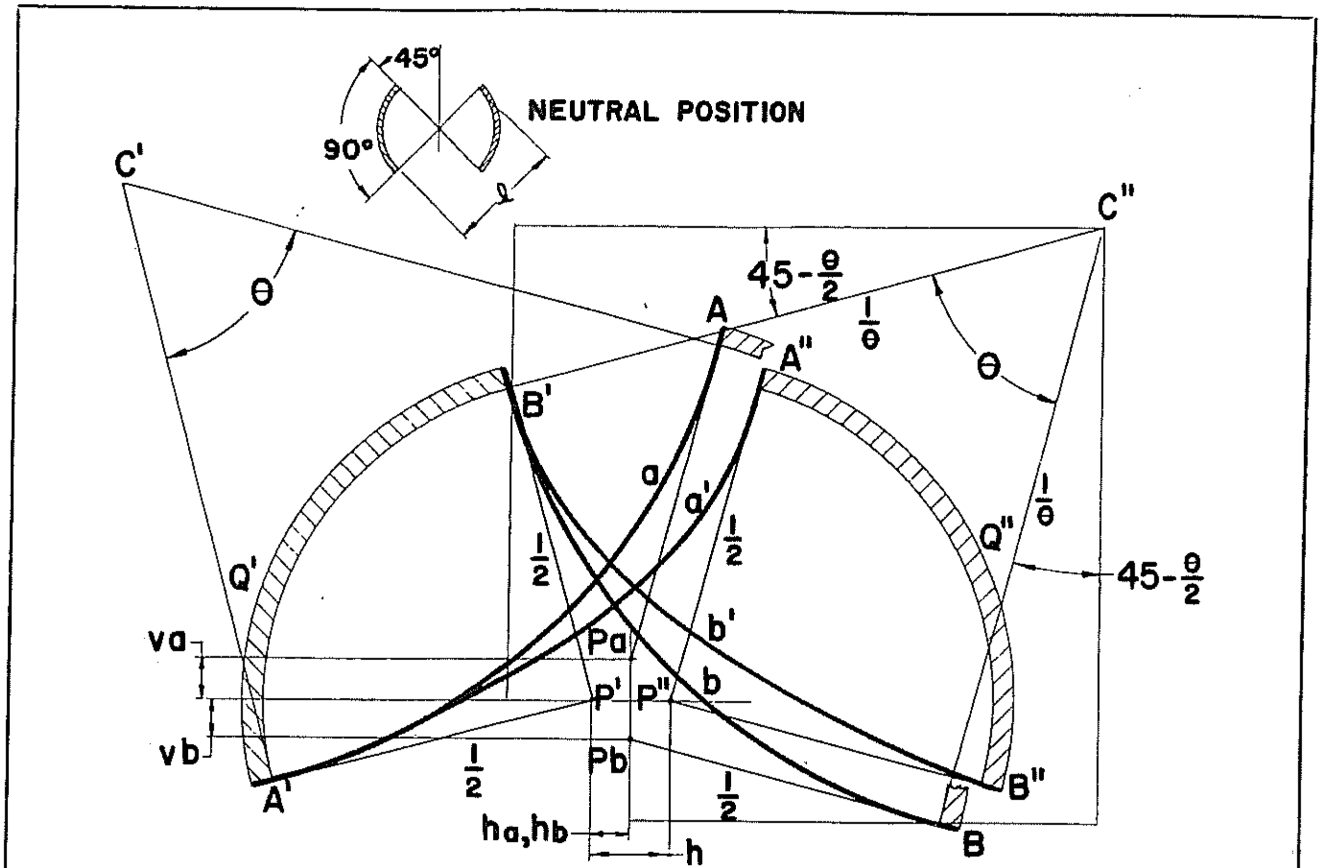

Fig. 5. Shift of geometric center with rotation.

The geometric center will be defined as the intersection of the tangents to the neutral planes of the flexures at their movable joints. It will coincide with the intersection of the flexure planes for an undeflected pivot and will be the center of the movable cylinder for a cylindrical pivot. The position of the geometric center is pertinent to the determination of instantaneous position of the movable member. Center shift will be defined as the distance between the geometric centers of the neutral and deflected positions.

The instantaneous center is the axis about which the moving member instantaneously rotates. The path of the instantaneous center is the centrode. It is pertinent to the kinematics of the moving member.

It is helpful to visualization of how the flexures provide constraint to first assume each flexure as being independently deflected by equal pure couples. In Figure 5, flexures of unit length have equal couples applied at A and B without the intermediate constraint of quadrant Q". The flexures have uniform width and thickness and hence will bend into equal circular arcs a and b. The ends of each flexure have been shown deflected one half the total pivot angular deflection in opposite directions from the neutral position. P_a would be the center of quadrant Q" if flexure b were omitted and P_b would be the center if flexure a were omitted. The horizontal and vertical displacements, h_b and v_b, of P_b from P′ will be seen to be.

$$V_b = h_b = \frac{1}{\theta}\cos\left(45^\circ - \frac{\theta}{2}\right) - \frac{\theta}{1}\sin\left(45^\circ - \frac{\theta}{2}\right)$$
$$- \frac{1}{2}\operatorname{Cos}\left(45^\circ - \frac{\theta}{2}\right) - \frac{1}{2}\sin\left(45^\circ - \frac{\theta}{2}\right)$$

(for θ measured in radians)
which reduces to:

$$V_b = h_b = \frac{\sqrt{2}}{2}\left(\frac{2\sin\frac{\theta}{2}}{\theta} - \cos\frac{\theta}{2}\right)$$

Similarly determining the horizontal and vertical displacements, h_a and V_a, of P_a from P′ will result in the same solution:

$$V_a = h_a = \frac{\sqrt{2}}{2}\left(\frac{2\sin\frac{\theta}{2}}{\theta} - \cos\frac{\theta}{2}\right)$$

If both flexures are constrained by quadrant Q" so points P_a and P_b are superimposed by translation only of A to A" and B to B", from symmetry, it is apparent that they (P_a and P_b) will lie on the horizontal axis and the circular arcs a and b will be changed to two other forms, a", and b", that will be mirror images of each other. The resulting deformation of arc a to a′ and arc b to b′ will increase the flexure curvature at the A" end and decrease the curvature at the B" end. Assume, in order to avoid the complexity of an exact mathematical proof, that the chordal lengths of arc a, a", arc b and b" will be equal. This would require the paths of translation of A to A" and B to B" to occur along circular arcs of radius equal to the chordal lengths with centers at A′ and B′. But the chords of the circular arcs a and b are 45° to the axis of symmetry and the distances A-A" and B-B" are small compared to the chordal lengths. Hence, the translation of A to A" and B to B" may be considered, with very small positive error, to travel along paths inclined 45° to the axis of symmetry.

Points P_a and P_b will similarly translate to coincidence at point P′. Therefore the shift of geometric center will be:

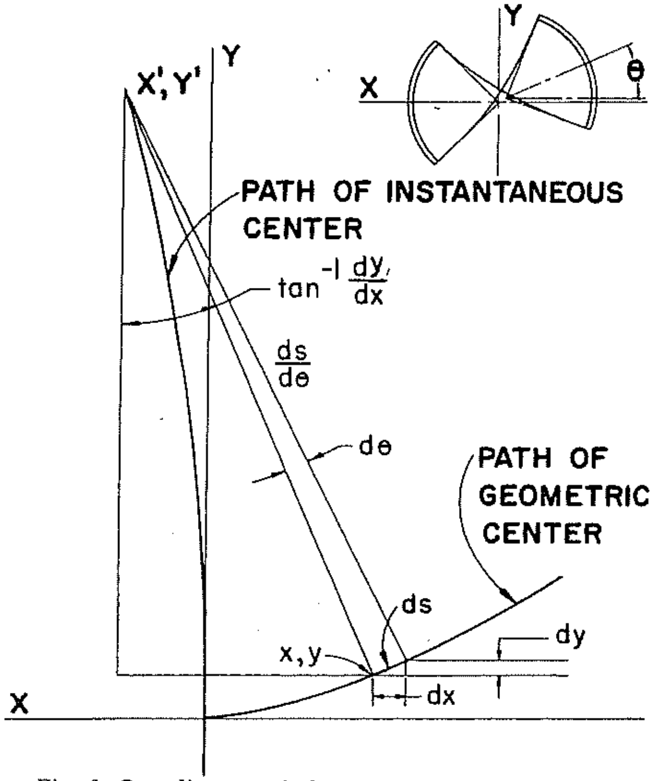

Fig. 6. Coordinates of the geometric center.

$h = 2h_s = 2h_a = \sqrt{2}\left(\frac{2\sin\frac{\theta}{2}}{\theta}\right) - \cos\frac{\theta}{2}$ (approximately)

In order to permit solution for small values of θ it is desirable to substitute the series form for the trigonometric functions, which results in:

$$h = \sqrt{2}\left(\frac{\theta^2}{2\cdot 3\,!}\ \frac{2\theta^4}{2^3\cdot 5\,!} + \frac{3\theta^6}{2^5\cdot 7\,!}\ \ldots\ldots\right)$$

which converges so rapidly for applicable values of θ that dropping all terms of the series except the first will result in a computation error less than 0.7% at 30° and less than 0.02% at 15°. Thus:

$$h = \frac{\sqrt{2}}{12}\theta^2 \text{ (very nearly)}$$

Applying this value of h and observing from symmetry that the geometric center must lie on a line passing through P' and displaced the angle, $\frac{\theta}{2}$, from the centerline of the fixed quadrant, the coordinates of the geometric center will be, as in Figure 6,:

$$x = h\cos\frac{\theta}{2} = \frac{\sqrt{2}}{12}\theta^2\cos\frac{\theta}{2}$$

$$y = h\sin\frac{\theta}{2} = \frac{\sqrt{2}}{12}\theta^2\sin\frac{\theta}{2}$$

The instantaneous center for a given deflection will lie on a normal, at the corresponding geometric center, to the path of the geometric center. The distance (of the instantaneous center) from the geometric center will be the derivative $\frac{ds}{d\theta}$, where ds is the differential distance along the path of the geometric center. An equation for the path of the instantaneous center may therefore be established as follows:

$$\frac{dx}{d\theta} = \frac{\sqrt{2}}{12}\left(2\theta\cos\frac{\theta}{2} - \frac{1}{2}\theta^2\sin\frac{\theta}{2}\right)$$

$$\frac{dy}{d\theta} = \frac{\sqrt{2}}{12}\left(2\theta\sin\frac{\theta}{2} - \frac{1}{2}\theta^2\cos\frac{\theta}{2}\right)$$

from which:

$$\frac{dy}{dx} = \frac{4\sin\frac{\theta}{2} + \theta\cos\frac{\theta}{2}}{4\cos\frac{\theta}{2} - \theta\sin\frac{\theta}{2}}$$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

or:

$$\frac{ds}{d\theta} = \frac{\sqrt{2}}{12}\theta\sqrt{\left(2\cos\frac{\theta}{2} - \frac{1}{2}\theta\sin\frac{\theta}{2}\right)^2 + \left(2\sin\frac{\theta}{2} + \frac{1}{2}\theta\cos\frac{\theta}{2}\right)^2}$$

which reduces to:

$$\frac{ds}{d\theta} = \frac{\sqrt{2}}{12}\theta\sqrt{4 + \frac{1}{4}\theta^2}$$

The coordinates of the instantaneous center will be:

$$x^1 = x - \frac{ds}{d\theta}\sin\tan^{-1}\frac{dy}{dx}$$

$$y^1 = y + \frac{ds}{d\theta}\cos\tan^{-1}\frac{dy}{dx}$$

which reduces to the following parametric equations:

$$x^1 = \frac{\sqrt{2}}{24}\theta\left(\theta\cos\frac{\theta}{2} - 4\sin\frac{\theta}{2}\right)$$

$$y^1 = \frac{\sqrt{2}}{24}\theta\left(\theta\sin\frac{\theta}{2} + 4\cos\frac{\theta}{2}\right)$$

By substituting the series forms for the sine and cosine and retaining only the first term of the resultant series the following equations for the instantaneous center are obtained:

$$x^1 = -\frac{\sqrt{2}}{24}\theta^2$$

$$y^1 = \frac{\sqrt{2}}{6}\theta$$

The preceding equations are based upon unit flexure strength. For standard cylindrical flexural pivots 1 = 0.6 D. Converting θ radians to θ degrees ($\theta°$) and substituting, the following equations are obtained for standard cylindrical flexural pivots of diameter D:

$h = 2.15\cdot 10^{-5}\,\theta^{\circ 2}\,D$ = displacement of geometric center

$x = 2.15\cdot 10^{-5}\,\theta^{\circ 2}\,D\cos\frac{\theta}{2}$
$y = 2.15\cdot 10^{-5}\,\theta^{\circ 2}\,D\sin\frac{\theta}{2}$
} Coordinates of geometric center

$x^1 = -1.07\cdot 10^{-5}\,\theta^{\circ 2}\,D$
$y^1 = .00247\,\theta°\,D$
} Coordinates of instantaneous center

The values of these coordinates are plotted on log-log graph, Figure 7.

The positions of the geometric and instantaneous centers thus obtained are nearly correct only for the condition of an applied couple. A normally negligible error will be introduced for translational forces small compared to pivot radial load capacity. Appreciable variation of the centers will occur for translational forces large compared to the radial load capacity. These forces may reduce the displacement of the geometric and instantaneous centers if the flexures are selectively orientated. Other supplementary forces are usually present in actual applications. Therefore, because of the complexity of a mathematical analysis for all conditions of loading, determination of characteristics is best accomplished by tests. These may be conducted in the mechanism or in specialized test equipment capable of duplicating the actual modes of loading where these can be defined.

Torsional spring rate: The torsional spring rate is:

$$\frac{M}{\theta} = \frac{EI}{l} = \frac{1}{12}\ \frac{Ewt^3}{l}$$

and the flexural stress will be:

$$S = \frac{1}{2}\ \frac{Mt}{I} = \frac{1}{2}\ \frac{Et\theta}{l}$$

where: M = applied couple
θ = deflection in radians
E = modulus of elasticity of flexure material
w = total of flexure widths
t = flexure thickness
l = flexure length
s = stress

Standard cylindrical flexural pivots with AISI 420 flexures are based on the following dimensions:

l = 0.6D
w = 1.36D
s = 137,000 psi
E = 29,400,000 psi

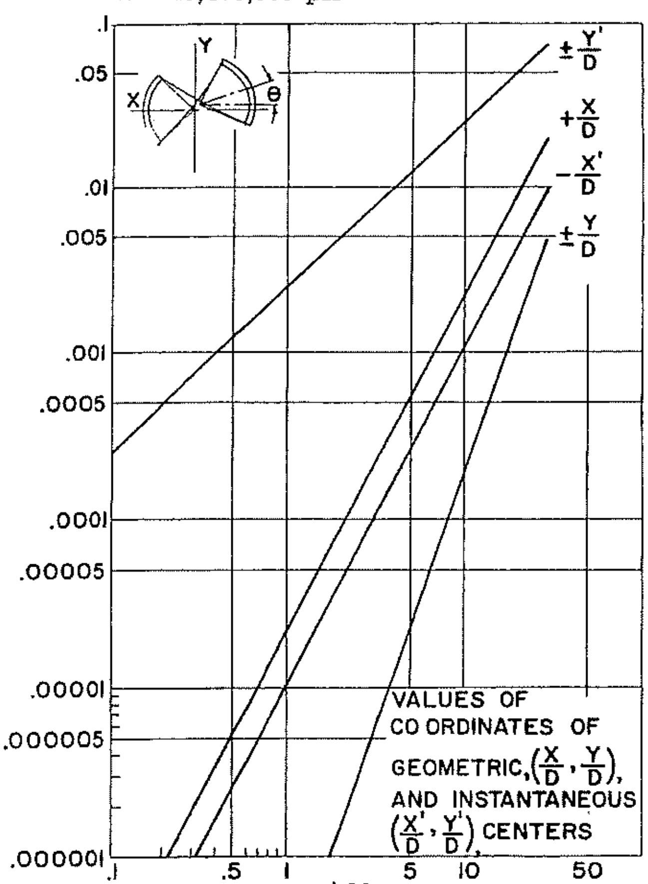

Fig. 7. Values of coordinates of geometric and instantaneous centers plotted on a log-log graph.

for which conditions only:

$$\frac{M}{\theta} = 187\left(\frac{10\ D}{\theta^\circ\ max}\right)^3 \text{ lb. in./radian}$$

Where: D = diameter of standard cylindrical pivot in inches

where θ° max = maximum rated deflection in degrees

Non-standard torsional spring rates may be provided by changing dimensions. Change in flexure thickness is the most economical modification as it involves only nominal expense for the rolling of special stock thickness, adjustment of machining operations and special handling. Other changes require special tooling.

Tests of a sample lot of 25 production pivots showed that the following distribution of torsional spring rate of .250 diameter, plus or minus 15° maximum travel pivots may be expected for a large group:

68% will fall within ± 3.5% of the mean value
94% will fall within ± 7% of the mean value
99.78% will fall within 10.5% of the mean value

Because the torsional spring rate varies as the cube of the flexure thickness and tolerances for very thin stock are proportionately larger than for thicker stock, greater variations in rate may be expected for small, low spring rate pivots than for larger high spring rate pivots.

Angular travel: Permissible angular travel is dependent upon tolerable center shift, hysteresis, fatigue life and load. Standard cylindrical pivots have permissible travel angles as high as plus and minus 30°.

Linearity: The torsional spring rate would be constant for an applied couple and a flexure material that ideally conforms to Young's Law if the flexures deformed into circular arcs. In practice, variations from linearity will occur due to distortion, translational loading and unavailability of a perfect material. Nevertheless, for travel angles less than ± 10°, errors due to non-linearity will be smaller than those due to friction in conventional bearings providing radial loads are conservative and flexure stresses do not exceed the proportional limit. Many measurements of linearity have indicated that errors in sensitive test equipment are greater than the deviation from linearity. Improved test equipment is required to provide reliable quantitative resolution of this characteristic.

Hysteresis: If a spring or flexure is cyclically deflected between two positions, the restoring force at any position will vary with the direction of approach to that position. The variation is called hysteresis. A graph of force plotted against position for cyclical deflection will be a closed loop called a hysteresis loop. For an ideal elastic material this loop would have no width and would be a straight line. Within certain limits some real materials approach this ideal.

Flexural pivot hysteresis is important in measuring devices in which the pivot provides the prime reactive force to a force to be measured or in devices in which zero or constant force is desired at a null position. Hysteresis may be relatively unimportant in applications where the applied forces are large compared to hysteresis forces.

A convenient numerical representation of hysteresis is the variation in null (zero restoring force) position as determined by approaching null from opposite extremes of deflection. On this basis, some typical measured values of hysteresis for standard cylindrical flexural pivots of AISI 420 material having a maximum angular travel rating of plus and minus 15 degrees are shown in Table 1.

Hysteresis value may be determined with a collimator which, because of its small range, of the order 16 minutes of arc, permits measurement at only the null position or any one position where the load conditions can be precisely duplicated.

Radial and axial load capacity: The only conditions that permit establishing simple radial load criteria are those which apply solely tensile or compressive stresses to the flexure in the neutral position, providing joint strength is not the limiting factor. The permissible radial load for a flexure stressed in tension will be:

$$P_t = wts$$

where: w = width of flexure to which load is applied
t = thickness of flexure
s = permissible tensile stress

The permissible stress for standard cylindrical flexural pivots has empirically been established as 109,000 psi to provide a load rating applicable to a deflected pivot without damaging it.

The radial load rating for cantilever type standard cylindrical flexural pivots with load applied at the center of a sleeve and with the end flexure in tension will be:

$$P_t = 12{,}000 \frac{D^2}{\theta^\circ \max} \text{ pounds}$$

The radial load rating for double end type pivots centrally loaded and either central or end flexures in tension will be:

$$P_t = 24{,}000 \frac{D^2}{\theta^\circ \max} \text{ pounds}$$

Though pivots will safely withstand loads as determined by these formulae, lower working loads will normally be dictated by requirements of radial stiffness, linearity of torsional spring rate and fatigue life.

For radial loads which place the flexures in compression, buckling will limit the permissible radial load.

$$P_c = \frac{4\pi^2 EI}{l^2}$$

is the formula for buckling of a column clamped at both ends and compressively loaded. If standard cylindrical pivot proportions are substituted the following formula results:

$$P_c = 3.04 \times 10^6 \frac{D^2}{(\theta \max)^3} \text{ pounds theoretical}$$

Tests of production pivots show that this formula results in values greater than those producing failure. On the basis of tests the constant in this formula is reduced so:

$$P_c = 1.8 \times 10^6 \frac{D^2}{(\theta^\circ \max)^3} \text{ pounds}$$

for standard cylindrical flexural pivots with radial load applied so as to compress the flexure.

A few tests of axial or thrust loads indicate a thrust capacity 2.5 times the rated tensile radial load capacity, P_t.

Radial spring rate: Computation of radial spring rate is mathematically complex and dependent upon many factors. Tests of radial spring rate of undeflected cantilever pivots with the radial load applied to the center of one sleeve, in a plane bisecting the plane of the flexures so as to apply tensile flexure loading, results in the following empirical formula:

$$\text{radial spring rate} = K = 2.6 \times 10^8 \frac{D^2}{(\theta^\circ \max)^3} \text{ pounds/inch}$$

Tests to be run are expected to show a higher value of the constant for double-end type pivots.

Fatigue life: Fatigue testing equipment which magnetically cycles flexural pivots at low sonic frequencies has been built. Fatigue testing is now in progress. Several pivots have concluded fatigue testing for which results are tabulated below. These were production pivots haveing a plus or minus 15° maximum travel rating and 0.1875, 0.2500 and 0.375 diameters. No radial load was applied.

Deflection Cycle	No. of Cycles	Result
± 15°	73,500	Flexure fatigued
± 15°	37,200	Flexure fatigued
± 11.25°	286,000	Flexure fatigued
± 11.25°	413,000	Flexure fatigued
± 11.25°	10,620,000	Flexure fatigued
± 11.25°	257,000	Flexure fatigued
± 7.5°	30,979,800	Failed after 6,600,000 additional cycles at ± 11°
± 7.5°	42,125,400	Flexure fatigued
± 7.5°	33,871,500	Failed after 37,584 additional cycles at ±11.1°
± 7.3°	83,975,000	No failure

The normally accepted theory for fatigue life of steels implies that a stress range applied for 30,000,000 cycles without failure is sufficient to indicate an indefinitely much larger number of cycles at the same or lower stress range may be applied without failure. On this basis, the limited fatigue testing to date would appear to indicate that the standard flexural pivots will withstand for an indefinitely large number of cycles angular deflections of slightly less than one half their maximum rated angular travel. ■ ■

TABLE I

TYPICAL HYSTERESIS VALUES FOR STANDARD PRODUCTION CYLINDRICAL FLEXURAL PIVOTS

Flexure Material: AISI 420 heat treated to R. 48/50

All Pivtos: rated maximum angular travel: ±15°

Pivot Diameter	Total Variation in null degrees after rotating pivot plus and minus: 2°	3.75°	7.5°	11.25°	15°
.1875	.0011	.025	.115	.312	.525
		.012	0.75	.250	.450
.2500		.000	.050	.120	.290
		.000	.012	.025	.112
		.001	.025	.062	.150
.3750		.000	.025	.050	.100
		.000	.025	.100	.250

References:

1. Eastman, Fred S., "The Unique Properties of Flexure Pivots," The Trend in Engineering, Vol. 12, No. 1 (Janury 1960), University of Washington.
2. Eastman, Fred S., "Flexure Pivots to Replace Knife Edges and Ball Bearings," University of Washington Experiment Station Bulletin No. 86, Nov., 1935.
3. Haringx, J. A., "The Cross Spring Pivot as a Constructional Element," App. Sci. Res., A, Vol. 1, No. 4, 1949.
4. Nickols, L. W., "The Design Characteristics of Cross Spring Pivots," The Engineer, Oct 12, 1951.
5. Wittrick, W. H., "The Theory of Symmetrical Crossed Flexure Pivots," Austral. Jour. Sci. Res., A, Vol. 1, No. 2, 1948.
6. Wittrick, W. H., "The Properties of Crossed Flexural Pivots, and the Influence of the Points at which the Strips Cross," Aero. Quar., Vol. II, Feb. 1951.
7. Young, W. E., "An Investigation of the Cross-Spring Pivot," Jour. App. Mech., June, 1944.